

Thermodynamics of a Schwarzschild Black Hole in Phantom Cosmology with Entropy Corrections

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Abstract: Motivated by some earlier works [1, 2] dealing with the study of generalized second law (GSL) of thermodynamics for a system comprising of a Schwarzschild black accreting a test non-self-gravitating fluid namely phantom energy in FRW universe, we extend them when the entropy of horizons of black hole and the cosmological undergo quantum corrections. Two types of such corrections are relevant here including logarithmic and power-law, while both are motivated from different theoretical backgrounds. We obtain general mathematical conditions for the validity of GSL in each case. Further we find that GSL restricts the mass of black hole for accretion of phantom energy. As such we obtain upper bounds on the mass of black hole above which the black hole cannot accrete the phantom fluid, otherwise the GSL is violated.

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I. INTRODUCTION

Modern approaches to unify theories of quantum mechanics and general relativity, for instance, string theory and loop quantum gravity predict that a black hole emits thermal radiations whose thermal spectrum might deviate from Planck black body spectrum at Planck scale [3]. The black hole's event horizon possesses temperature inversely proportional to black hole mass and with an entropy proportional to its horizon's surface area (in units $c = G = \hbar = 1$) i.e.

$$S_h = \frac{A_h}{4}, \quad (1)$$

where $A_h = 4\pi R_h^2$ is the area of the black hole's event horizon. Therefore the horizon entropy (1) becomes

$$S_h = \pi R_h^2. \quad (2)$$

These seminal connections between black holes and thermodynamics were initially made by Hawking and Bekenstein several decades ago [4]. The Hawking temperature and horizon entropy together with the black hole mass obey the first law of thermodynamics $TdS = dE + pdV$. Padmanabhan showed that Einstein field equations for a spherically symmetric spacetime can be recast in the form of first law of thermodynamics [5]. Cai & Kim applied the similar formalism and demonstrated that by applying the first law of thermodynamics to the apparent horizon of a Friedmann-Robertson-Walker universe the Friedmann equations of the universe with any spatial curvature can be derived from the first law [6]. Recently similar results have been obtained in scalar tensor, Gauss-Bonnet, Lovelock and $f(R)$ gravities by different authors [7]. Jacobson showed that Einstein field equation is nothing but an equation of state of spacetime i.e. the Einstein equation can be derived by assuming the universality of (1) on any local Rindler horizons [8]. In black hole physics, the generalized second law is a conjecture about black hole thermodynamics which states that “the sum of the black hole entropy (1/4 of the horizon area) and the common (ordinary) entropy in the black hole exterior never decreases’ originally proposed by Bekenstein [9, 10]. However as discussed by Jacobson [8], the entire framework of black hole thermodynamics and, in particular, the notion of black hole entropy extends to any causal horizon. In cosmological spacetime, the corresponding object is *apparent horizon* [12].

The power-law correction to entropy which appear in dealing with the entanglement of quantum fields in and out the horizon is given by is [11]

$$S_h = \frac{A_h}{4} \left(1 - K_\alpha A_h^{1-\frac{\alpha}{2}} \right), \quad (3)$$

where α is a dimensionless constant and $A_h = 4\pi R_h^2$ is the area while R_h is the radius of the horizon.

$$K_\alpha = \frac{\alpha(4\pi)^{\frac{\alpha}{2}-1}}{(4-\alpha)r_c^{2-\alpha}}. \quad (4)$$

where r_c is a cross-over scale, R_h is the radius and A_h is area of the cosmological horizon. For entropy to be a well-defined quantity, we require $\alpha > 0$. The second term in (3) can be regarded as a power-law correction to the area law, resulting from entanglement, when the wave-function of the field is chosen to be a superposition of ground state and excited state [13]. Several aspects of power-law corrected entropy (3) have been studied in literature including GSL [14], power-law entropy corrected models of dark energy [15].

The quantum corrections provided to the entropy-area relationship leads to the curvature correction in the Einstein-Hilbert action and vice versa. The logarithmic corrected entropy is [16]

$$S_h = \frac{A_h}{4} + \beta \log\left(\frac{A_h}{4}\right) + \gamma. \quad (5)$$

These corrections arise in the black hole entropy in loop quantum gravity due to thermal equilibrium fluctuations and quantum fluctuations. Jamil & Sadjadi [17] showed that in a (super) accelerated universe GSL is valid whenever $\beta(<) > 0$ leading to a (negative) positive contribution from logarithmic correction to the entropy. In the case of super acceleration the temperature of the dark energy is obtained to be less or equal to the Hawking temperature. Using the corrected entropy-area relation motivated by the loop quantum gravity, Karami et al [18] investigated the validity of the GSL in the FRW universe filled with an interacting viscous dark energy with dark matter and radiation. They showed that GSL is always satisfied throughout the history of the universe for any spatial curvature regardless of the dark energy model.

We consider a scenario of a spatially flat, homogeneous, isotropic universe filled with phantom energy and contain a Schwarzschild black hole. Since this simple cosmic system consists of three components, we associate the entropy with each component. The entropy of Schwarzschild black hole and FRW universe is proportional to the size (area) of their horizon (only if the entropic corrections are ignored) while for phantom energy, the entropy is calculated via the first law of thermodynamics. We assume that the black hole accretes phantom energy such that the mass of black hole decreases very slowly while preserving the spherical symmetry. This kind of accretion is termed as quasi-static accretion, and the corresponding black hole in quasi-static state (i.e. Schwarzschild geometry is still valid) [19]. While accretion, the entropies of phantom energy and the black hole vary, but the total entropy of the system remains non-decreasing. From the argument

of first law of thermodynamics, we notice that the rate of change of entropy of phantom energy $\dot{S}_d = T^{-1} \dot{H} R_h^2$ depends on its temperature T and the rate of change of Hubble parameter \dot{H} . The entropy $\dot{S}_d > 0$ if both $T > 0 (< 0)$ and $\dot{H} > 0 (< 0)$. According to some earlier thermodynamic approaches to gravity, the form of entropy-area relation applies the same to different horizons, here we apply the same principle in different sections. First we choose the classical Bekenstein-Hawking relation for entropy for the horizons of both black hole and cosmological and investigate the GSL. Later we study the same phenomenon by including the power-law and logarithmic corrections to the entropy of both black hole and cosmological horizon. Although the horizon of a Schwarzschild black hole is uniquely defined, the form of cosmological horizon is not so. We use the two well-known forms of cosmic horizons i.e. the future event horizon and the apparent horizon (both will be defined later). The idea of the combined effect of a cosmological system involving a dark energy component and a black hole has also been explored in ‘entropic cosmology’ [20]. In these works, Cai and collaborators investigated the dynamical thermal balance of a double-screen model (corresponding to cosmic horizon and black hole horizon) which can realize both inflation and late time acceleration of the Universe.

Earlier Izquierdo & Pavon [1] investigated the GSL for a system comprising of a Schwarzschild black accreting phantom energy in FRW universe. They showed that GSL is violated. Later Sadjadi [2] investigated the same problem and showed that GSL will be satisfied if the temperature is not taken as de Sitter temperature. Understanding the evolution of a black hole in a FRW cosmological background is a very old problem starting from Hawking & Carr [21] whose satisfactory resolution is still not available, however several approximations (like the present analysis) are available in the literature, e.g. [1, 2, 22].

The plan of our paper is as follows: In section-II, we write down the basic equations of standard model of cosmology and the definition of generalized second law of thermodynamics in the present context. In sections-III, IV and V, we study the GSL with Bekenstein-Hawking entropy-area relation, power-law entropy correction and logarithmic entropic correction, respectively with the use of apparent and event horizons. In section-VI, we discuss the constraints imposed by GSL on the black hole mass for accretion of phantom energy. In last section, we write down the conclusion giving a summary of our results.

II. BASIC EQUATIONS

Assuming homogeneous, isotropic and spatially flat Friedmann-Robertson-Walker metric:

$$ds^2 = -dt^2 + a(t)^2(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)). \quad (6)$$

The Friedmann equations are

$$H^2 = \frac{8\pi}{3}\rho, \quad (7)$$

$$\dot{H} = -4\pi(\rho + p). \quad (8)$$

The continuity equation is

$$\dot{\rho} + 3H(\rho + p) = 0, \quad (9)$$

where ρ and p are the energy density and pressure of phantom energy. Assuming the phantom energy as a perfect fluid, we specify it by a phenomenological equation of state

$$p = w\rho, \quad (10)$$

where $w < -1$ is the dimensionless state parameter of phantom energy. The notion of phantom energy was introduced by Caldwell et al [23] as a separate candidate of explaining cosmic acceleration. The phantom energy possesses some esoteric properties: “Phantom energy rips apart the Milky Way, solar system, Earth, and ultimately the molecules, atoms, nuclei, and nucleons of which we are composed, before the death of the Universe in a Big Rip” [23]. Although the model of phantom energy is consistent with the observational data [24], the cosmic doomsday (or Big Rip) can be avoided in certain theories of modified gravity [25]. Thermodynamical studies show that phantom energy possesses negative temperature and positive entropy [26]. However some discussion contrary to [26] on phantom thermodynamics has been performed in [27]. Nojiri et al discussed the occurrence of different types of future singularities in phantom cosmology [28]. One of these singularities is Big Rip described as: (or Type-I): As $t \rightarrow t_s$, $a \rightarrow \infty$, $\rho \rightarrow \infty$, $|p| \rightarrow \infty$. A form of scale factor satisfying these conditions can be $a(t) = a_0(t_s - t)^n$, $t_s > t$, $n = \frac{2}{3(1+w)} < 0$, $a_0 > 0$ where t_s is the Big Rip time. The corresponding Hubble parameter goes like

$$H(t) = \frac{2}{3(1+w)(t - t_s)}. \quad (w < -1) \quad (11)$$

Thus Hubble parameter diverges as $t \rightarrow t_s$.

Babichev et al [29] demonstrated that the accretion of phantom energy as a test perfect fluid on a stationary spherically symmetric black hole gradually reduces the mass of black hole. Further the mass of black holes approach to zero near the time called Big Rip. They deduced that the rate of change in black hole mass due to phantom energy accretion goes like [29]

$$\dot{M} = 4\pi A r_h^2 (\rho + p) < 0. \quad (12)$$

Above A is a positive dimensional constant, r_h is the Schwarzschild radius, while energy density and pressure of phantom energy violates the null energy condition $\rho + p < 0$. Later on their analysis extended in various ways: using bulk viscosity, generalized Chaplygin gas, Riessner-Nordstrom and Kerr-Newmann black holes [31] to list a few.

Making use of (8) in (12), we can write

$$\dot{M} = -4AM^2\dot{H}. \quad (13)$$

On integration, we obtain

$$M(H) = \frac{1}{C_1 + 4AH}, \quad (14)$$

where C_1 is a constant of integration. From (14), we observe that mass of black hole decreases as the rate of cosmic expansion increases.

For generalization of these results to many black holes, one can follow the procedure of Khan and Israel [30] by first replacing the spherically symmetric black hole by a point mass located on the z -axis. Then we can use the superposition principle to write the general expression of gravitational potential of the system. In this case, we can replace the mass of a BH by the total mass of many point particles. However we are not interested to such extensions and beyond the scope of our paper.

To calculate the entropy of phantom fluid, we use the first law of thermodynamics which relates the pressure, energy density, total energy and temperature of phantom energy i.e.

$$dS_d = \frac{1}{T}(dE + pdV) = \frac{1}{T}((\rho + p)dV + Vd\rho). \quad (15)$$

The form of GSL containing the time derivatives of entropies of black hole's horizon, phantom energy and cosmological's horizon is

$$\dot{S}_{tot} \equiv \dot{S}_{BH} + \dot{S}_d + \dot{S}_A \geq 0. \quad (16)$$

Above \dot{S}_{tot} is the rate of change of total entropy which must be non-decreasing. Here we would like to comment that as a result of accretion, the dark energy goes inside the BH event horizon.

Since the major bulk of dark energy density lies outside the BH horizon than to its interior, we do not associate entropy to DE lying inside the BH horizon. In other words, the entropy of DE is solely determined from its quantity contained between the cosmic and BH horizons while the entropy of DE inside BH horizon is ignored. Also note that in a DE filled Universe, we assume that the interior of BH horizon is always filled with DE as a result of accretion.

The analysis in later sections is based on the assumption of thermal equilibrium: the temperature of black hole's event horizon, cosmological horizon and the phantom energy are the same. But this assumption in cosmological setting is very ideal, since major components of the universe including dark matter, dark energy and radiation (CMB and neutrinos inclusive) have entirely different temperatures [32]. But Karami and Ghaffari [33] recently demonstrated that the contribution of the heat flow between dark energy and dark matter for GSL in non-equilibrium thermodynamics is very small, $O(10^{-7})$. Therefore the equilibrium thermodynamics is still preserved. Further, if there is any thermal difference in the fluid and the horizons, the transfer of energy across the horizons might change the geometry of horizons [17].

III. GSL WITH BEKENSTEIN-HAWKING ENTROPY

A. Use of Event Horizon

The *future event horizon* is the distance that light travels from present time till infinity is defined as

$$R_E(t) = a(t) \int_t^\infty \frac{dt'}{a(t')} < \infty, \quad (17)$$

whose time derivative is

$$\dot{R}_E = HR_E - 1. \quad (18)$$

The temperature of future event horizon is proportional to de Sitter's universe horizon

$$T_h = \frac{bH}{2\pi}, \quad (19)$$

where b is a constant. Depending on the argument chosen from thermodynamics of phantom energy [26, 27], b can be positive or negative. Integrating (18) and using (11), we obtain a time evolution of R_E :

$$R_E = C_2(t - t_s)^{\frac{2}{3(1+w)}} - \frac{3(t - t_s)(1 + w)}{1 + 3w}, \quad (20)$$

where C_2 is a constant of integration. In the present context, Sadjadi [2] studied the cosmological thermodynamics using the cosmic future event horizon R_E . However in his detailed analysis, the author ignored a very important first term on right hand side of (20), which will change significantly the results for the validity of GSL.

B. Use of Apparent Horizon

The *apparent horizon* is a null surface with vanishing expansion or the boundary surface of anti-trapped region [12]. In a spatially flat FRW universe, the apparent horizon is $R_A = H^{-1}$ (also called Hubble horizon). This horizon is consistent if we insist on the validity of holography during inflation i.e. the apparent horizon is the holographic boundary of the FRW universe. The temperature of apparent horizon is the same as temperature of a de Sitter's universe horizon [6]

$$T_h = \frac{H}{2\pi} \quad (21)$$

Using (16) the form of GSL at the apparent horizon becomes

$$\dot{S}_{tot} = -32\pi A \dot{H} M^3 \geq 0. \quad (22)$$

In terms of Hubble parameter alone,

$$\dot{S}_{tot} = \frac{-32\pi A \dot{H}}{(C_1 + 4AH)^3} \geq 0. \quad (23)$$

In the above case (23), mathematically GSL will hold under two situations: Case-I: (1) $\dot{H} \leq 0$, $C_1 + 4AH > 0$ or Case-II: (1) $\dot{H} \geq 0$, (2) $C_1 + 4AH < 0$. However physically, under phantom dominated era, only case-II is relevant. Thus the apparent horizon expands in the phantom phase while the future event horizon contracts.

IV. GSL WITH POWER-LAW ENTROPY CORRECTION

We extend our previous study here by taking into account the correction to horizon's entropy of the power-law form.

A. Use of Event Horizon

Using the definition of GSL (16) gives

$$\begin{aligned}\dot{S}_{tot} = & 2\pi R_E \dot{R}_E \left[1 - \frac{\alpha(4\pi)^{\frac{\alpha}{2}-1}}{2r_c^{2-\alpha}} \left(\pi R_E^2 \right)^{1-\frac{\alpha}{2}} \right] \\ & - 32\pi A \dot{H} M^3 \left[1 - \frac{\alpha(4\pi)^{\frac{\alpha}{2}-1}}{2r_c^{2-\alpha}} \left(\frac{4\pi}{M^2} \right)^{1-\frac{\alpha}{2}} \right] \\ & + \frac{2\pi \dot{H}}{bH} \dot{R}_E \geq 0.\end{aligned}\quad (24)$$

Its easy to interpret the positivity of (24) by writing the above equation as a total derivative form

$$\begin{aligned}\dot{S}_{tot} = & \frac{d}{dt} \left[\pi R_E^2 - \frac{\pi\alpha}{4-\alpha} (2r_c)^{\alpha-2} R_E^{4-\alpha} \right. \\ & \left. + 4\pi M^2 - \frac{8\pi}{2r_c^{2-\alpha}} M^\alpha \right] \\ & + \frac{2\pi \dot{H}}{bH} \dot{R}_E \geq 0.\end{aligned}\quad (25)$$

For phantom $b < 0$, and in both cases $\dot{H} < (> 0)$, $\dot{R}_E < (> 0)$, the general condition to satisfy GSL is

$$\begin{aligned}& \frac{d}{dt} \left[\pi R_E^2 - \frac{\pi\alpha}{4-\alpha} (2r_c)^{\alpha-2} R_E^{4-\alpha} \right. \\ & \left. + 4\pi M^2 - \frac{8\pi}{2r_c^{2-\alpha}} M^\alpha \right] \geq 0.\end{aligned}\quad (26)$$

B. Use of Apparent Horizon

Using the definition of GSL (16) gives

$$\begin{aligned}\dot{S}_{tot} = & 2\pi \frac{\dot{H}}{H^3} \frac{\alpha(4\pi)^{\frac{\alpha}{2}-1}}{2r_c^{2-\alpha}} \left(\frac{\pi}{H^2} \right)^{1-\frac{\alpha}{2}} - 32\pi A \dot{H} M^3 \\ & \times \left[1 - \frac{\alpha(4\pi)^{\frac{\alpha}{2}-1}}{2r_c^{2-\alpha}} \left(\frac{\pi}{M^2} \right)^{1-\frac{\alpha}{2}} \right] \geq 0.\end{aligned}\quad (27)$$

We rewrite it in the following total derivative form

$$\begin{aligned}\dot{S}_{tot} = & \frac{d}{dt} \left[\frac{\pi\alpha}{\alpha-4} \frac{(2r_c)^{\alpha-2}}{H^{4-\alpha}} \right. \\ & \left. + 4\pi M^2 - 4\pi (2r_c)^{\alpha-2} M^\alpha \right] \geq 0.\end{aligned}\quad (28)$$

V. GSL WITH LOGARITHMIC ENTROPY CORRECTION

We extend our previous study here by taking into account the correction to horizon's entropy of the logarithmic form.

A. Case of Event Horizon

Using the definition of GSL (16) gives

$$\begin{aligned}\dot{S}_{tot} = & 2\dot{R}_E\left(\pi R_E + \frac{\beta}{R_E}\right) - 8A\dot{H}M\left(\beta - 4\pi M^2\right) \\ & + \frac{2\pi\dot{H}}{bH}\dot{R}_E \geq 0.\end{aligned}\quad (29)$$

For phantom $b < 0$, and in both cases $\dot{H} < (> 0)$, $\dot{R}_E < (> 0)$, the general condition to satisfy GSL is

$$\begin{aligned}\dot{S}_{tot} = & \frac{d}{dt}\left[\pi R_E^2 + 2\beta \log(MR_E) - 4\pi M^2\right] \\ & + \frac{2\pi\dot{H}}{bH}\dot{R}_E \geq 0.\end{aligned}\quad (30)$$

B. Case of Apparent Horizon

Using the definition of GSL (16) gives

$$\dot{S}_{tot} = -2\pi\frac{\dot{H}}{H}\beta M - 8A\dot{H}M^2\left(\beta - 4\pi M^2\right) \geq 0. \quad (31)$$

Writing the above equation as a total derivative form, we get

$$\dot{S}_{tot} = \frac{d}{dt}\left(\beta \log\left(\frac{M}{H}\right) - 2\pi M^2\right) + \frac{2\pi\dot{H}}{bH}\dot{R}_E \geq 0. \quad (32)$$

VI. PHANTOM ENERGY ACCRETION BY BLACK HOLE: GSL CONSTRAINTS

Pacheco & Horvath [34] investigated the generalized second law of thermodynamics for a static spherically symmetric black hole accreting phantom energy. They showed that for a phantom fluid violating the null energy condition ($\rho + p < 0$), the Euler relation ($\rho + p = TS$) and Gibbs relation ($E + pV = TS$, assuming $E = \rho V$ we get $(\rho + p)V = TS$) allows two different possibilities for the entropy and temperature of phantom energy: a situation when the entropy is negative and the temperature is positive or vice versa. In the former case, if GSL is valid, the accretion of phantom energy is not allowed while in the later case, there is a critical black hole mass above which the accretion process is not allowed. In another study, Lima et al [35] discussed the thermodynamics of phantom energy with chemical potential μ_0 and found the EoS parameter of the form

$$w \geq -1 + \frac{\mu_0 n_0}{\rho_0}, \quad (33)$$

where n_0 is the number density and ρ_0 is the energy density of phantom energy. The authors deduced if $\mu_0 < 0$ then w describes a phantom fluid. Lima and co-workers [36] showed that the temperature of the phantom fluid without chemical potential is positive definite but entropy is negative $S_d < 0$. But their claim is problematic since if considering the usual statistical definition of entropy, than phantom energy must not exist at all. Later Pacheco [37] ruled out the previous results of phantom accretion by black holes with chemical potential [35, 36]. Below, we adapt the procedure of [34] about constraints imposed by GSL on the mass of a black hole. We find that there exists a critical value of black hole mass (or the upper bound) M_c below which accretion of phantom is allowed. Note that in the subsequent analysis, we deal phantom energy without chemical potential on account of [37]. Also the foregoing analysis with Bekenstein-Hawking entropy has been done in [34], therefore, we will continue our work with logarithmic and power-law entropy corrections only.

A. Using Logarithmic Entropy

We consider the new entropy as a sum of black hole entropy and entropy of phantom energy as

$$S_n = f(X) + \kappa \rho^{\frac{1}{1+w}} V, \quad (34)$$

where κ is a constant and $f(X)$ is a given function of area. We first start with logarithmic entropy

$$f(X) = X + \beta \log(X) + \gamma, \quad X \equiv \frac{A}{4} = 4\pi M^2. \quad (35)$$

On account of accretion, the change in the entropy of black hole and the phantom energy is

$$\Delta S_n = f'(X) \Delta X + \frac{\kappa}{1+w} \rho^{\frac{-w}{1+w}} V \Delta \rho. \quad (36)$$

The total energy conservation for this system is [34]

$$\Delta M = -\frac{1}{2}(1+w)V\Delta\rho, \quad (37)$$

or

$$V\Delta\rho = -\frac{2\Delta M}{1+w}. \quad (38)$$

Using (38) in (36), we get

$$\Delta S_n = \left[8\pi M f'(X) - \frac{2\kappa}{(1+w)^2} \rho^{\frac{-w}{1+w}} \right] \Delta M. \quad (39)$$

Since mass of black hole decreases due to accretion of phantom energy i.e. $\Delta M < 0$, we require

$$\Delta S_n > 0 \Rightarrow 4\pi M_c f'(X_c) - \frac{\kappa}{(1+w)^2} \rho^{\frac{-w}{1+w}} < 0. \quad (40)$$

From (35), we have $f'(X) = 1 + \frac{\beta}{X}$. Thus

$$M_c^2 - \left(\frac{\kappa}{4\pi(1+w)^2} \rho^{\frac{-w}{1+w}} \right) M_c + \frac{\beta}{4\pi} \geq 0. \quad (41)$$

The discriminant of (41) is

$$\delta = \left(\frac{\kappa}{8\pi(1+w)^2} \rho^{\frac{-w}{1+w}} \right)^2 - \frac{\beta}{4\pi} \geq 0. \quad (42)$$

Here two cases are possible: (1) $\delta = 0$ gives $(M_c - M)^2 < 0$ which is nonphysical and mathematically not possible, (2) $\delta > 0$ implies $(M_c - M_+)(M_c - M_-) > 0$, where M_{\pm} are the roots of (41) given by

$$M_{\pm} = \frac{\kappa}{8\pi(1+w)^2} \rho^{\frac{-w}{1+w}} \pm \sqrt{\left(\frac{\kappa}{8\pi(1+w)^2} \rho^{\frac{-w}{1+w}} \right)^2 - \frac{\beta}{4\pi}}, \quad (43)$$

while the critical black hole mass lies in the range

$$M_- < M_c < M_+, \quad (44)$$

where M_+ and M_- are the corresponding upper and lower bounds on critical mass of black hole.

B. Using Power-Law Corrected Entropy

The form of power-law corrected entropy is

$$S(X) = X[1 - K_{\alpha}(4X)^{1-\frac{\alpha}{2}}]. \quad (45)$$

Here condition (39) implies

$$\begin{aligned} & 8\pi M_c \left[1 - K_{\alpha} \left(2 - \frac{\alpha}{2} \right) (16\pi)^{1-\frac{\alpha}{2}} M_c^{2-\alpha} \right] \\ & - \frac{2\kappa}{(1+w)^2} \rho^{\frac{-w}{1+w}} < 0. \end{aligned} \quad (46)$$

In terms of r_c , the general equation for critical mass to satisfy is

$$4\pi M_c \left[1 - \frac{\alpha}{2} \left(\frac{2M_c}{r_c} \right)^{2-\alpha} \right] - \frac{\kappa}{(1+w)^2} \rho^{\frac{-w}{1+w}} < 0. \quad (47)$$

To solve (47), we consider some special cases of (47) for different values of $\alpha = 1, 2, 3, 4, 5$.

C. $\alpha = 1$

We can write (47) in terms of cross-over scale parameter r_c for convenience:

$$M_c - \frac{\alpha}{r_c} M_c^2 - \frac{\kappa}{4\pi(1+w)^2} \rho^{-\frac{w}{1+w}} < 0. \quad (48)$$

Here discriminant of the above expression is

$$\delta = 1 - \frac{\kappa}{\pi r_c (1+w)^2} \rho^{-\frac{w}{1+w}} \geq 0. \quad (49)$$

The corresponding roots are

$$M_{\pm} = \frac{r_c}{2\alpha} (1 \pm \sqrt{\delta}). \quad (50)$$

The critical black hole mass lies in the range

$$M_- < M_c < M_+, \quad (51)$$

where M_+ and M_- are the corresponding upper and lower bounds on critical mass of black hole.

D. $\alpha = 2$

In this case, we get the lower bound on the mass of black hole:

$$M > M_c = \frac{-\kappa}{4\pi(1+w)^2} \rho^{\frac{-w}{1+w}}, \quad (52)$$

while $\kappa < 0$ since mass can not be negative physically.

E. $\alpha = 3$

Here we obtain an upper bound on the mass of black hole as

$$M < M_c = \frac{3}{4} r_c + \frac{\kappa}{4\pi(1+w)^2} \rho^{\frac{-w}{1+w}}. \quad (53)$$

F. $\alpha = 4$

The equation to be satisfied by critical mass is

$$M_c^2 - \frac{\kappa}{4\pi(1+w)^2} \rho^{\frac{-w}{1+w}} M_c - \frac{r_c^2}{2} < 0 \quad (54)$$

The discriminant of the above equation is

$$\delta = \left(\frac{\kappa \rho^{\frac{-w}{1+w}}}{4\pi(1+w)^2} \right)^2 + 2r_c^2 > 0, \quad (55)$$

while the roots are

$$M_{\pm} = \frac{\kappa}{8\pi(1+w)^2} \rho^{\frac{-w}{1+w}} \pm \frac{\sqrt{\delta}}{2}. \quad (56)$$

Further critical mass satisfies $M_- < M_c < M_+$.

G. $\alpha = 5$

Here we have the following cubic equation

$$M_c^3 - A_1 M_c^2 - B_1 < 0, \quad (57)$$

where

$$A_1 = \frac{\kappa}{4\pi(1+w)^2} \rho^{-\frac{w}{1+w}}, \quad B_1 = \frac{5}{16} r_c^3 \quad (58)$$

We define a new variable $y = M + \frac{A_1}{3}$, than the (57) converts to

$$y^3 + py + q < 0, \quad p = -\frac{A_1^2}{3}, \quad q = B_1 + \frac{2}{27} A_1^3 \quad (59)$$

Since $p < 0$, this cubic equation has three distinct solutions which are

$$y_1 = \sqrt{-\frac{p}{3}} \cos(\psi), \quad (60)$$

$$y_2 = \sqrt{-\frac{p}{3}} \cos\left(\frac{\psi}{3} + \frac{\pi}{3}\right), \quad (61)$$

$$y_3 = \sqrt{-\frac{p}{3}} \cos\left(\frac{\pi}{3} - \frac{\psi}{3}\right), \quad (62)$$

where

$$\cos(\psi) = \frac{108B_1 + 8A_1^3}{8A_1^3}.$$

Thus there are two possibilities for the value of critical mass

$$M_c < M_1, \quad M_2 < M_c < M_3, \quad (63)$$

where

$$M_1 = \frac{\kappa}{6\pi(1+w)^2} \rho^{-\frac{w}{1+w}} \left(-\frac{1}{2} + \cos(\psi) \right), \quad (64)$$

$$M_2 = \frac{\kappa}{6\pi(1+w)^2} \rho^{-\frac{w}{1+w}} \left(-\frac{1}{2} + \cos\left(\frac{\pi}{3} - \frac{\psi}{3}\right) \right), \quad (65)$$

$$M_3 = \frac{\kappa}{6\pi(1+w)^2} \rho^{-\frac{w}{1+w}} \left(-\frac{1}{2} + \cos\left(\frac{\pi}{3} + \frac{\psi}{3}\right) \right). \quad (66)$$

VII. CONCLUSION

In this paper, we studied the generalized second law of thermodynamics for a system comprising of a Schwarzschild black accreting a test non-self-gravitating fluid namely phantom energy in FRW universe. We are interested if the entropy of this whole system is positive or not. Since second law of thermodynamics is fundamental law of physics, its validity needs to be checked for a thermal system. As is known the form of entropy-area relation for any causal horizon (either black hole or cosmological) is the same, we discussed entropies of these horizons with classical relation of Hawking and with quantum corrections, for the sake of consistency. However the entropy of phantom energy was calculated using first law of thermodynamics. Using two different forms of cosmological horizon, we found general conditions for the validity of GSL in the present context. Next we used the GSL to impose some restrictions (upper or lower bounds) on the mass of black hole under which a black hole can accrete phantom energy. In this article, we are unable to predict whether the observable Universe is dominated by phantom energy or not. Rather we assumed that it contains phantom energy as claimed by Caldwell [23, 24]. Moreover so far there are no theoretical constraints on the model parameters (like β and γ etc) except the BH mass coming from our model.

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